BRST cohomologies for non-critical strings

2000 J. Phys. A: Math. Gen. 33 7773

(http://iopscience.iop.org/0305-4470/33/43/308)

J. Phys. A: Math. Gen. **33** (2000) 7773–7791. Printed in the UK PII: S0305-4470(00)15543-X

BRST cohomologies for non-critical strings

Zbigniew Hasiewicz
Institute of Theoretical Physics, University in Białystok, ul. Lipowa 41, 15-424 Białystok, Poland
E-mail: zhas@physics.uwb.edu.pl

Received 17 July 2000

Abstract. The BRST cohomologies of non-critical massive string models are analysed in detail in the range of dimensions $1 < d < 26$. The classical theorems on vanishing of Dolbeaut type and relative cohomologies are proved. The no-ghost theorem is proved in the same range of dimensions. It is also shown that the BRST complex of the massive string model is a resolution of the non-critical Nambu–Goto string in all spacetime dimensions $1 < d < 25$.

Introduction

This paper is devoted to a detailed study of the quantum BRST cohomologies of the non-critical string model formulated in [1]. The model has many promising features. It provides a consistent relativistic quantum theory of a one-dimensional extended object in subcritical dimensions $2 \leqslant d \leqslant 25$. It was shown [1, 2] within the framework of the 'old covariant' quantization scheme that it is equivalent to a non-critical Nambu–Goto string [3] with longitudinal degrees of freedom [4] but admits a much more tractable (gauge-equivalent) dual picture in the Fock space generated by transverse and Liouville degrees of freedom. It was also demonstrated [2] that the model can be *locally* described within the framework of a quantum lightcone gauge [3] which, in turn, allowed for an effective analysis of its spin content. The mass and spin spectra, although unacceptable for fundamental string theory, are quite interesting from the point of view of the original motivations which introduced the quantum relativistic string into physics [5]. The absence of massless excited states in their spectra ($d = 25$ is the unique exception) indicates that it is not hopeless to expect that they may give a proper description of the states of low-energy QCD. The main problem which must be solved is a consistent theory of their interactions.

There is a very attractive idea of joining–splitting interactions [6] which cannot be pursued, however, at least in its pure form, in the case of massive strings. The Mandelstam interaction vertex, taken in its raw form, is simply not relativistic invariant in subcritical dimensions. What is even more important is that the microcausal structure of lightcone graphs does not agree with what is known about the classical motions of massive strings. It was shown in [1] that the worldsheets of non-critical strings are generically not timelike and that the microcausality condition imposed on the classical solutions reduces the system to a Nambu–Goto string model.

It seems that in order to construct the interactions of massive strings, a better understanding of their quantum geometry is an inevitable condition.

The main motivation of the author to investigate the BRST cohomologies of non-critical strings was to make a small step towards a consistent theory of their interactions. Although quantum interactions of critical string theory were originally constructed within the framework of the so-called 'old covariant' formalism of dual theory [5] and the lightcone formulation of Mandelstam [6], the role of the BRST formulation in the subsequent development of the theory is by all means of primary importance [7].

The problem of the BRST description of non-critical strings was first raised in [8]. It was shown there that there exists a quantum complex corresponding to the non-critical Polyakov string in the canonical formulation [9]. Almost ten years later the statement on vanishing of BRST cohomologies of the complex constructed in [8] was implicitly 'spelled out' [10] on the ground of the results obtained and the methods used in [11].

It is shown in this paper that the BRST complex constructed for the non-critical string model of [1] is a resolution [12, 13] of the non-critical Nambu–Goto string theory. The method of obtaining this result is a bit different from that commonly used in the literature [11, 14]. As an intermediate step towards computation of the cohomology spaces bigraded complexes analogous to those of Dolbeaut complexes of complex geometry [12, 13] are introduced and an analogue of the Dolbeaut–Grotendieck lemma is proved. The vanishing theorem for relative cohomologies is then obtained as a direct conclusion. A proof of the no-ghost theorem based on a comparison of the Euler–Poincaré characteristic of the complex with its signature [15, 16] is given. It is also shown that the gauge equivalence in the sense of the 'old covariant' formalism† translates directly into the BRST cohomological equivalence in the bigraded complex.

The classical model of non-critical string is defined by the functional [1]:

$$S[M, g, \varphi, x] = -\frac{\alpha}{2\pi}\int_M \sqrt{-g}\,\mathrm{d}^2z\, g^{ab}\partial_a x^\mu \partial_b x_\mu - \frac{\beta}{2\pi}\int_M \sqrt{-g}\,\mathrm{d}^2z\,\left(g^{ab}\partial_a\varphi\partial_b\varphi + 2R_g\varphi\right) \tag{1}$$

which is an extension of the standard d-dimensional string worldsheet action by the Liouville action for an additional scalar field. The detailed analysis [1] of the variational problem for (1) leads, in the conformal gauge, to the constrained phase space system which is most conveniently parametrized by real canonical pairs (x^μ, p_μ); $\mu = 0, \ldots, d-1$ to describe centre-of-mass motions and complex variables $a_m^\mu,\ u_m;\ m \in \mathbb{Z}\setminus\{0\}$ constructed out of higher Fourier modes of the real maps x^μ and φ and their canonically conjugated momenta. The zero modes of the Liouville field are eliminated by the superselection rule imposed by the consistency condition [1] of the variational problem for (1).

The Poisson brackets of the canonical variables read as

$$\{p_\mu, x^\nu\} = \eta^{\mu\nu} \qquad \{a_m^\mu, a_n^\nu\} = \mathrm{i}m\eta^{\mu\nu}\delta_{m+m} \qquad \{u_m, u_n\} = \mathrm{i}m\delta_{m+m}. \tag{2}$$

The classical constrains

$$L_m = \tfrac{1}{2}\sum_{n\in\mathbb{Z}} a_{-n}a_{m+n} + \tfrac{1}{2}\sum_{n\in\mathbb{Z}} u_{-n}u_{m+n} + 2\sqrt{\beta}\,\mathrm{i}mu_m \tag{3}$$

are of mixed type

$$\{L_m, L_n\} = \mathrm{i}(m-n)L_{m+n} - 4\mathrm{i}\beta m^3\delta_{m+n} \tag{4}$$

where L_0 is a unique one of first class.

The classical string model is thus a canonical system with constraints of mixed type. For this reason the application of the standard cohomological quantization method in this case is not so well geometrically justified as for first-class constrained systems. Nevertheless, the quantum model of a non-critical string admits, as will be shown, a consistent description in terms of the cohomology classes of the standard BRST complex.

† The states are defined as the classes modulo null vectors.

1. The quantum string model and BRST complexes

The first quantization of the classical model is performed by constructing an irreducible representation space for the Poisson algebra of canonical variables (2). This space is defined as a direct integral:

$$\mathcal{H} = \int \mathrm{d}^d p\, \mathcal{H}(p) \tag{5}$$

of pseudo-unitary Fock modules. Every $\mathcal{H}(p)$ is generated by the algebra of string excitation operators:

$$\begin{aligned} &[a_m^\mu, a_n^\nu] = m\eta^{\mu\nu}\delta_{m+n} \\ &[u_m, u_n] = m\delta_{m+n} \qquad m, n \in \mathbb{Z} \qquad \mu, \nu = 0, \ldots, d-1 \end{aligned} \tag{6}$$

out of the vacuum vector $\omega(p)$ satisfying $a_m^\mu\omega(p) = 0 = u_m\omega(p); m > 0$. The vacuum vectors are distributional eigenfunctions for the momentum operators: $P^\mu\omega(p) = p^\mu\omega(p)$ and are formally normalized by the condition $\big(\omega(p), \omega(p')\big) = \delta(p - p')$.

The properties above together with formal conjugation rules $a_m^{\mu *} = a_{-m}^\mu$, $u_m^* = u_{-m}$ determine the unique scalar product in (5).

The operators corresponding to the classical constrains (3) are given by the standard normally ordered expressions:

$$L_m = \sum_{k\in\mathbb{Z}} : a_{m+k}a_{-k} : + \sum_{k\in\mathbb{Z}} : u_{m+k}u_{-k} : + 2\mathrm{i}\sqrt{\beta} m u_m + 2\beta\delta_{m0} \qquad m \in \mathbb{Z}. \tag{7}$$

The central term in their structural relations:

$$[L_m, L_n] = (m-n)L_{m+n} + \tfrac{1}{12}m\left(m^2 - 1\right)(d + 1 + 48\beta)\,\delta_{m+n} \tag{8}$$

is modified with respect to that of (4) by the normal ordering anomaly.

It is important for further constructions that the spaces $\mathcal{H}(p)$ of (5) are decomposable into a direct sum of finite-dimensional eigensubspaces of the string level operator $R_{\mathrm{str}} = L_0 - (1/2\alpha)p^2 - 2\beta$:

$$\mathcal{H}(p) = \bigoplus_{N_{\mathrm{str}}\geqslant 0} \mathcal{H}^{N_{\mathrm{str}}}(p). \tag{9}$$

Recall that within the framework of the 'old covariant' quantization method the physical subspace of (5) is defined as an invariant subspace

$$\mathcal{H}_{\mathrm{phys}} = \{\Psi;\ (L_m - a\delta_{m0})\,\Psi = 0\} \tag{10}$$

with respect to the maximal isotropic subalgebra of (8). A free real parameter a defines the beginning of the string mass spectrum and is left to be fixed by natural consistency conditions: the unitarity which is necessary for quantum mechanical interpretation and relativistic invariance. The unitarity condition means that the inner product in the space above should be non-negative. Then the quotient of (10) by the subspace of null vectors gives (after completion) the Hilbert space of states.

Within the framework of the BRST formalism the construction above is replaced by (and should, in fact, be equivalent to) a cohomological description of physical degrees of freedom of the constrained quantum system. In mathematical language, the BRST complex should be a resolution [12, 13] of the space of physical states.

The ghost sector $\mathcal{C}_\infty$† of the string BRST complex is generated by the family of ghost $\{c_m\}_{m\in\mathbb{Z}}$ and antighost $\{b_m\}_{m\in\mathbb{Z}}$ modes with the usual anti commutation relations

$$\{b_m, c_n\} = \delta_{m+n} \qquad \{b_m, b_n\} = 0 \qquad \{c_m, c_n\} = 0. \tag{11}$$

The vacuum of the ghost space is defined as the unique vector ω satisfying

$$c_m\omega = 0 \quad m > 0 \qquad \text{and} \qquad b_m\omega = 0 \quad m \geqslant 0. \tag{12}$$

The non-degenerate scalar product in $\mathcal{C}_\infty$ is fixed by the normalization condition $(\omega, c_0\omega) = 1$ of the ghost vacuum, the commutation relations (11) and formal conjugation properties $b_m^* = b_{-m}$, $c_m^* = c_{-m}$ imposed on ghost modes.

The representation of the constraint algebra in $\mathcal{C}_\infty$ is realized by the operators constructed in such a way that under the commutator the anti-ghost modes $\{b_m\}$ are in an adjoint representation of the Virasoro algebra, while the ghosts $\{c_{-m}\}$ are in a coadjoint one. This natural principle‡ is supplemented by the rule that the operators act on the vacuum vector which amounts to the normal ordering prescription for the ghost modes:

$$:c_m b_k: = \begin{cases} c_m b_k & m < 0 \\ -b_k c_m & m > 0 \\ \frac{1}{2}(c_0 b_0 - b_0 c_0) & m = k = 0 \end{cases}$$

and results in $\mathcal{L}_m = \sum_{k\in\mathbb{Z}} (k-m) : c_{-k} b_{m+k}:$ with

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n} - \delta_{m+n}\left(\tfrac{26}{12}(m^3 - m) + 2m\right). \tag{13}$$

The differential in $\mathcal{C}_\infty$ is defined to be

$$d_\infty = \frac{1}{2}\sum_{m>0} c_{-m}\mathcal{L}_m + \frac{1}{2}c_0\mathcal{L}_0 + \frac{1}{2}\sum_{m>0}\mathcal{L}_{-m}c_m. \tag{14}$$

Because of the anomaly in (13) it neither commutes with Virasoro operators ($\mathcal{L}_0$ being the unique exception) nor it is nilpotent:

$$d_\infty^2 = -\sum_{m>0}\left(2m + \tfrac{26}{12}m(m^2-1)\right) c_{-m}c_m. \tag{15}$$

The ghost level operator $\mathcal{L}_0 = \sum m : c_{-m} b_m :$ is diagonalizable in $\mathcal{C}_\infty$ and its commutator with a differential does vanish. Therefore, the space $\mathcal{C}_\infty$ can be decomposed into a direct sum of finite-dimensional and d_∞ invariant subspaces of fixed ghost level

$$\mathcal{C}_\infty = \bigoplus_{N\geqslant 0} \mathcal{C}_\infty^N. \tag{16}$$

The whole space $\mathcal{C}_\infty$ as well as every subspace of fixed level is graded by half-integral eigenvalues of the ghost number operator:

$$G = \sum_{m\in\mathbb{Z}} : c_{-m}b_m: =: G_0 + \tfrac{1}{2}(c_0b_0 - b_0c_0). \tag{17}$$

The ghost number operator $G_0 = \sum_{m>0}(b_{-m}c_m - c_{-m}b_m)$ with an integral spectrum is introduced here for later convenience in dealing with the so-called relative complex.

† This notation is used in order to stress the isomorphism of the ghost sector with the space of semi-infinite forms of [17]. The vacuum of the ghost sector is then identified with the (semi-)infinite wedge product $\omega \sim c_1c_2\ldots$.
‡ This is the method of construction of an equivariant complex for any Lie group [18].

The ghost numbers r admissible at level N are bounded by the condition $N \geqslant \frac{1}{2}(r^2 - \frac{1}{4})$ and the scalar product in $\mathcal{C}_\infty$ which pairs the spaces of opposite ghost numbers is non-degenerate on the subspaces (16) of level decomposition.

The total space of the string BRST complex is constructed as an appropriate tensor product of the covariant string space $\mathcal{H}$ (5) and the ghost sector $\mathcal{C}_\infty$. It is defined as a direct sum of subspaces of fixed total level:

$$\mathcal{C} = \bigoplus_{N \geqslant 0} \int \mathrm{d}^d p\, \mathcal{C}^N(p) \qquad \text{where} \quad \mathcal{C}^N(p) = \bigoplus_{N_{\mathrm{str}} + N_{\mathrm{gh}} = N} \mathcal{H}^{N_{\mathrm{str}}}(p) \otimes \mathcal{C}_\infty^{N_{\mathrm{gh}}}. \tag{18}$$

The differential in this complex is defined by the standard [18] formulae:

$$D = \sum_{m \in \mathbb{Z}} (L_m - \delta_{m0} a) \otimes c_{-m} + 1 \otimes d_\infty. \tag{19}$$

It is not nilpotent:

$$D^2 = \tfrac{1}{12} \sum_{m > 0} c_{-m} c_m \left(m(m^2 - 1)(d + 1 + 48\beta - 26) - 24m(1 - a) \right) \tag{20}$$

unless the central terms of (8) and of (13) cancel each other, i.e. the free parameters of the model take their critical values [8]:

$$\beta = \frac{25 - d}{48} \qquad \text{and} \qquad a = 1. \tag{21}$$

The differential D commutes with momentum operators P_μ as well as with the total level operator $R^{\mathrm{tot}} = \mathcal{L}_0 + L_0 - \frac{1}{2} P^2 - 2\beta$. The complex (18) then splits into a sum/integral of subcomplexes supported by the subspaces $\mathcal{C}^N(p)$.

The cohomology space of the total complex (18) may be formally reconstructed as a direct sum and direct integral out of cohomology spaces

$$H^r(p) := \frac{Z^r(p)}{B^r(p)} \qquad Z^r(p) := \ker D|_{\mathcal{C}^r(p)} \qquad B^r(p) = \mathrm{Im}\, D|_{\mathcal{C}^{r-1}(p)} \tag{22}$$

of the corresponding subcomplexes.

The label for the level number was suppressed in order to keep the amount of baroque style in the notation bounded at some reasonable level. It will always be assumed, however, that the spaces under consideration are of finite dimension. The convention to denote the spaces of cocycles by the root letter Z and those of coboundaries by the root letter B will be kept throughout this paper.

There is one more operator which commutes with D and can be diagonalized, namely $L_0^{\mathrm{tot}} = \mathcal{L}_0 + L_0 - 1 = \{b_0, D\}$. The elements from the kernel of L_0^{tot} are supported on the mass shells S_N:

$$p^2 = -m_N^2 \qquad \text{with} \quad m_N^2 = 2\alpha \left(N^{\mathrm{str}} + N^{\mathrm{gh}} - \frac{d - 1}{24} \right) \tag{23}$$

and are not normalizable with respect to the measure under the direct integral.

Since the kinetic operator L_0^{tot} is exact, the cohomology classes are, in fact, determined by the cochains in its kernel: if $D\Psi = 0$ and $L_0^{\mathrm{tot}}\Psi = \kappa\Psi$; $\kappa \neq 0$ then $\Psi = D(1/\kappa) b_0 \Psi$. For this reason, as far as only free theory is taken into account, it is possible to restrict the considerations to the on-mass-shell complex

$$\mathcal{C}_0 = \bigoplus_{N \geqslant 0} \int_{S_N} \mathrm{d}\mu^N(p)\, \mathcal{C}^N(p) \tag{24}$$

with momenta satisfying (23). The direct integrals over mass shells S_N are taken with respect to the unique Lorenz invariant measures $\mathrm{d}\mu^N(p)$.

It is worth stressing that all excited states are massive for spacetime dimension less than 25. In the case of $d = 25$ the situation is similar to that of the critical string model: the first excited level is massless and perfect vacuum states $p = 0$ without string excitations are admissible on-shell. These states, as will be seen later, are responsible for non-vanishing of higher cohomology classes.

Since the complex is restricted to on-mass-shell cochains and the ghost c_0 corresponding to the kinetic operator L_0^{tot} does not contribute to the mass spectrum it is natural to introduce a relative complex:

$$\mathcal{C}_{\mathrm{rel}}(p) := \{\Psi \in \mathcal{C}_0(p);\ b_0\Psi = 0\} \qquad \mathcal{C}_0(p) = \mathcal{C}_{\mathrm{rel}}(p) \oplus c_0\mathcal{C}_{\mathrm{rel}}(p) \tag{25}$$

with the differential D_{rel} given by the formulae:

$$D = D_{\mathrm{rel}} + L_0^{\mathrm{tot}}c_0 + Mb_0 \qquad M = -2\sum_{m>0} mc_{-m}c_m = \{D, c_0\}. \tag{26}$$

The relative differential is nilpotent on mass shell: $D_{\mathrm{rel}}^2 = -ML_0^{\mathrm{tot}}$. The ghost zero mode c_0 is absent in $\mathcal{C}_{\mathrm{rel}}$ and it is convenient to assume that this complex is graded by integral eigenvalues of shifted ghost number operator G_0 introduced in (17).

The relation between relative cohomology spaces

$$H_{\mathrm{rel}}^r(p) = \frac{Z_{\mathrm{rel}}^r(p)}{B_{\mathrm{rel}}^r(p)} \tag{27}$$

and those of absolute cohomology (22) is not straightforward in the general case. The problem of reconstruction of absolute cohomologies out of (27) will be solved after the relative cohomology classes are identified.

Note that the original scalar product induced on $\mathcal{C}(p)$ by (6), (11), (12) is zero when restricted to the relative complex. The non-degenerate pairing in $\mathcal{C}_{\mathrm{rel}}(p)$ is thus defined by

$$(\Psi, \Psi')_{\mathrm{rel}} := (\Psi, c_0\Psi') \qquad \Psi, \Psi' \in \mathcal{C}_{\mathrm{rel}}(p). \tag{28}$$

There are still two more complexes which are tightly related to string theory and were used in the formulation of a critical gauge field theory of strings [19]. Their importance was also indicated in [17] in a slightly wider context. If one agrees to interpret the BRST complex and its relative subcomplex as the counterparts of equivariant de Rham complexes [18] then the two correspond to a Dolbeaut complex of complex geometry and its Hermitian dual [13].

The decomposition of [19] of the differential D_{rel} corresponds, in fact, to the almost complex structure on the quotient space $Vir/\mathbb{C}L_0$ underlying the relative complex†. The extension of the complex structure to the space $\mathcal{C}_{\mathrm{rel}}$ is given by the operator

$$J = \exp\left(-\mathrm{i}\tfrac{1}{2}\pi\left(\sum_{m>0} c_{-m}b_m + \sum_{m>0} b_{-m}c_m\right)\right). \tag{29}$$

The decomposition of the ghost and anti-ghost spaces into eigenspaces of (29) induces a bigrading of the space (25) of relative cochains. The bihomogeneous components $\mathcal{C}_b^a(p)$ containing the elements of bidegree (a, b) are spanned by the vectors of the form

$$\Psi_b^a(p) = \upsilon \otimes c_{-j_1}\ldots c_{-j_a}b_{-i_1}\ldots b_{-i_b}\omega \qquad \upsilon \in \mathcal{H}(p). \tag{30}$$

† It is natural complex structure on $Diff(S^1)/S^1$ [20] described by the operator $J(\sum_{m\neq 0} x^m l_m) = \mathrm{i}\sum_{m\neq 0} \mathrm{sign}(m)\, x^m l_m$ at the tangent space at unity.

The almost complex structure (29) is integrable (also in the analytic sense as demonstrated in [20]) and this property amounts to the following decomposition of the relative differential $D_{\mathrm{rel}} = \mathcal{D} + \overline{\mathcal{D}}$:

$$\mathcal{D} = \sum_{m>0} L_{-m} \otimes c_m + \sum_{m>0} c_m \tau_{-m} + \partial \qquad \overline{\mathcal{D}} = \sum_{m>0} L_m \otimes c_{-m} + \sum_{m>0} c_{-m} \tau_m + \overline{\partial}. \tag{31}$$

The operators $\mathcal{D}$ and $\overline{\mathcal{D}}$ are mutually anti-adjoint, $\mathcal{D}^* = -\overline{\mathcal{D}}$, with respect to the scalar product (28) in the relative complex. As suggested by the notation above, the operators ∂ and $\overline{\partial}$ denote the canonical differentials of $Vir_{\mp}$ subalgebras of Vir:

$$\partial = -\tfrac{1}{2} \sum_{m,k>0} (m-k) b_{-k-m} c_k c_m \qquad \text{and} \qquad \overline{\partial} = -\tfrac{1}{2} \sum_{m,k>0} (m-k) c_{-m} c_{-k} b_{k+m}$$

while the operators

$$\tau_m = \sum_{k>m} (m+k) b_{m-k} c_k \qquad \text{and} \qquad \tau_{-m} = \sum_{k>m} (m+k) c_{-k} b_{k-m} \quad m > 0$$

realize the adjoint cross actions of respective subalgebras on themselves and define the induced representations of $Vir_{\mp}$ on the corresponding ghost subspaces.

The conditional on-shell nilpotency $D_{\mathrm{rel}}^2 = -ML_0^{\mathrm{tot}}$ is equivalent to

$$\mathcal{D}^2 = 0 \qquad \overline{\mathcal{D}}^2 = 0 \qquad \text{and} \qquad \mathcal{D}\overline{\mathcal{D}} + \mathcal{D}\overline{\mathcal{D}} = -ML_0^{\mathrm{tot}} \quad (= 0 \text{ on-shell}).$$

The complexes and cohomologies associated with the nilpotent operators $\mathcal{D}$ and $\overline{\mathcal{D}}$ will be defined in the next section.

2. Cohomologies of BRST complexes

This section is devoted to the computation of cohomology spaces associated with the original string complex (18). First, and it seems the most important one, is the complex of Dolbeaut type [13] and its dual. Both are supported by the same underlying space of cochains as that of (25) but are endowed with differentials given by (1, 0) and (0, −1) components $\overline{\mathcal{D}}$ and $\mathcal{D}$ of D_{rel} (31). Since $\overline{\mathcal{D}}$ ($\mathcal{D}$) preserves the anti-ghost (ghost) number, the relative complex decomposes into a direct sum of subcomplexes:

$$\mathcal{C}(p) = \bigoplus_{ab} \mathcal{C}_b^a(p) \qquad \overline{\mathcal{D}} : \mathcal{C}_b^a(p) \to \mathcal{C}_b^{a+1}(p) \qquad \mathcal{D} : \mathcal{C}_b^a(p) \to \mathcal{C}_{b-1}^a(p) \tag{32}$$

with fixed anti-ghost (ghost) number. The corresponding cohomology spaces are bigraded:

$$\overline{\mathcal{H}}_b^a(p) = \frac{\overline{\mathcal{Z}}_b^a(p)}{\overline{\mathcal{B}}_b^a(p)} \qquad \mathcal{H}_b^a(p) = \frac{\mathcal{Z}_b^a(p)}{\mathcal{B}_b^a(p)}. \tag{33}$$

Note that the arrow for $\mathcal{D}$ is reversed (32) with respect to that of classical complex cohomology theory [13]: $\overline{\mathcal{D}}$ raises the ghost number, while $\mathcal{D}$ lowers the anti-ghost degree. This is a reflection of the fact that the ghost sector stems from a semi-infinite vacuum form instead of a 0-form. The vacuum form when properly interpreted, is a section of a non-trivial holomorphic square root of a canonical bundle [21]. This is, on the one hand, the source of non-vanishing curvature (15) but, on the other hand, ensures that the mass spectrum is bounded from below and makes the theory acceptable from a physical point of view.

Both $\overline{\mathcal{D}}$ and $\mathcal{D}$ are nilpotent in an off-shell complex. Consequently, the cohomologies of these operators remain intact independently of whether the mass-shell condition is satisfied or not provided (as will be seen below) the momenta of the states are non-zero. This property indicates that bigraded complexes and bigraded cohomologies (33) may play a significant role in the formulation of non-critical string field theory. They will be useful for computation and identification of relative (27) and absolute (22) cohomologies too.

It will be shown first that there is an analogue of Poincaré–Serre duality for bigraded cohomologies (33).

Lemma 2.1 (Poincaré–Serre duality). *The spaces* $\overline{\mathcal{H}}^a_b(p)$, $\mathcal{H}^b_a(p)$ *are mutually dual with respect to*

$$\langle[\Psi], [\Psi']\rangle := (\Psi, \Psi')_{\mathrm{rel}} \qquad \Psi \in \overline{\mathcal{Z}}^a_b(p) \quad \Psi' \in \mathcal{Z}^b_a(p). \tag{34}$$

Proof. Choosing the representing spaces $\overline{\mathcal{R}}^a_b$ and $\mathcal{R}^b_a$ for cohomologies (33) it is possible to write the direct sum decompositions:

$$\overline{\mathcal{Z}}^a_b = \overline{\mathcal{R}}^a_b \oplus \overline{\mathcal{B}}^a_b \qquad \mathcal{Z}^b_a = \mathcal{R}^b_a \oplus \mathcal{B}^b_a$$

where $\overline{\mathcal{B}}^a_b$, $\mathcal{B}^a_b$ denote the subspaces of exact cocycles.

The differentials $\mathcal{D}$ and $\overline{\mathcal{D}}$ are mutually adjoint with respect to (28) and $(\overline{\mathcal{B}}^a_b, \mathcal{Z}^b_a)_{\mathrm{rel}} = 0 = (\mathcal{B}^b_a, \overline{\mathcal{Z}}^a_b)_{\mathrm{rel}}$. Hence the pairing $\langle\,,\,\rangle$ is well defined on cohomology classes.

It remains to show that it is non-degenerate. Choose any complementary spaces $\overline{\mathcal{Z}}'^a_b$, $\mathcal{Z}'^b_a$ such that

$$\mathcal{C}^a_b = \overline{\mathcal{R}}^a_b \oplus \overline{\mathcal{B}}^a_b \oplus \overline{\mathcal{Z}}'^a_b \qquad \mathcal{C}^b_a = \mathcal{R}^b_a \oplus \mathcal{B}^b_a \oplus \mathcal{Z}'^b_a.$$

Then $\overline{\mathcal{D}}$, $\mathcal{D}$ are injective on $\overline{\mathcal{Z}}'^a_b$ and $\mathcal{Z}'^b_a$, respectively. This together with the fact that $(\,,\,)_{\mathrm{rel}}$ pairs $\mathcal{C}^a_b$ with $\mathcal{C}^b_a$ in a non-degenerate way implies, in turn, that the spaces $\overline{\mathcal{B}}^a_b$, $\mathcal{Z}'^b_a$ and $\mathcal{B}^b_a$, $\overline{\mathcal{Z}}'^a_b$, respectively, are dual with respect to the scalar product. Hence the cohomology representing spaces $\overline{\mathcal{R}}^a_b$ and $\mathcal{R}^b_a$ must also be paired by $(\,,\,)_{\mathrm{rel}}$ in a non-degenerate way and are mutually dual. □

A similar reasoning yields the Poincaré duality for relative cohomology spaces: $H^r_{\mathrm{rel}} \simeq \left(H^{-r}_{\mathrm{rel}}\right)^*$ with respect to (28) and an analogous result for absolute cohomologies.

The universal and effective tool to compute the cohomologies (33) is the technique of spectral sequences [12, 24]. In the concrete context of this paper, however, the technology of spectral sequences and filtered complexes is not needed in its full generality. It is possible to adopt the general ideas of reasoning [12, 24] while keeping the considerations on elementary level as in [11, 25].

Two ways of proceeding are possible here. A more abstract one [16, 23] based on the vanishing theorem for Chevalley–Eilenberg homology [22] with values in a free module or that of [11] exploiting the same technology of filtered complexes, but more explicitly related to the kinematical situation under consideration. Since the first method is only seemingly more general than the second—both are, in fact, equivalent in the context of string theory—the steps of technical preparation to prove the Dolbeaut–Grotendieck vanishing lemma for (33) will follow those of [11].

It will be shown that Dolbeaut (by a slight abuse of the classic terminology) cohomologies (33) do vanish for positive ghost degrees $(a, *)$; $a > 0$ at any momentum $p \neq 0$.

Fix a non-zero momentum p. Then there exists an adapted lightcone basis† $\{k_\pm, e_i\}_{i=1}^{d-2}$ of the momentum space such that $p^+ := k_+ p \neq 0$. The Virasoro operators of the string sector

† Consisting of two light like vectors $k_\pm^2 = 0$, $k_+ k_- = -1$ and an orthonormal basis $\{e_i\}_{i=1}^{d-2}$ of Euclidean transverse space.

(7) written in the lightcone basis adapted to the momentum p take the form

$$L_m = -\frac{1}{\sqrt{\alpha}}p^+a_m^- + L_m^{\mathrm{tr}} + L_m^{\mathrm{Li}} - \sum_{\substack{n\neq 0\\ m+n\neq 0}} a_{n+m}^+ a_{-n}^- - \frac{1}{\sqrt{\alpha}}p^- a_m^+ \tag{35}$$

where L_m^{tr} and L_m^{Li} denote the operators given by standard expressions (7) in transverse and Liouville modes, respectively. Introduce a new gradation in the spaces (32) by assigning the following filtration degrees† to the elementary modes:

$$\begin{array}{ll} \deg(a_m^-) = -1 & \deg(b_m) = -1 \\ \deg(a_m^+) = +1 & \deg(c_m) = +1 \\ \deg(a_m^i) = 0 & \deg(h_m) = 0 \\ m \in \mathbb{Z}\setminus\{0\} & 1 \leqslant i \leqslant d-2. \end{array} \tag{36}$$

The spaces $\mathcal{C}_b(p)$ of the bigraded complex (33) decompose into direct sums of filtration homogeneous components: $\mathcal{C}_b(p) = \bigoplus_f \mathcal{C}_{b;f}(p)$. According to this decomposition the differential $\overline{\mathcal{D}}$ decays into three parts:

$$\begin{array}{ll} \overline{\mathcal{D}} = \overline{\mathcal{D}}_{(0)} + \overline{\mathcal{D}}_{(1)} + \overline{\mathcal{D}}_{(2)} & \overline{\mathcal{D}}_{(i)} : \mathcal{C}^a_{b;f}(p) \to \mathcal{C}^{a+1}_{b;f+i}(p) \\ \overline{\mathcal{D}}_{(0)} = -\sum\limits_{m>0} \dfrac{1}{\sqrt{\alpha}} p^+ a_m^- c_{-m}. & \end{array} \tag{37}$$

The operators $\overline{\mathcal{D}}_{(1)}$ and $\overline{\mathcal{D}}_{(2)}$ can be easily read off from (35) and (31) but their explicit form is not needed. Out of (37) only the component $\overline{\mathcal{D}}_{(0)}$ of filtration degree zero will be in use. It is nilpotent and defines the cohomology spaces $\overline{\mathcal{H}}^a_{b;f}(p)$ localized at fixed filtration degree f.

Lemma 2.2.

$$\overline{\mathcal{H}}^a_{b;f}(p) = 0 \qquad a > 0 \quad p \neq 0.$$

Proof. The operator $\mathcal{R} = \sum_{m>0}(mc_{-m}b_m - a^+_{-m}a^-_m)$ counting the level of c ghost and a^+ string excitations has filtration degree zero and is exact: $\mathcal{R} = \{\overline{\mathcal{D}}_{(0)}, \mathcal{K}\}$ with $\mathcal{K} = (\sqrt{\alpha}/p^+)\sum_{m>0} a^+_{-m}b_m$. Consequently, all $\overline{\mathcal{D}}_{(0)}$ closed states not in the kernel of $\mathcal{R}$ (in particular those with $a > 0$) are exact. □

Making use of this simple statement it is possible to prove the strict counterpart of the Dolbeaut–Grotendieck lemma of classical complex geometry [13] on vanishing of bigraded cohomologies (33).

Lemma 2.3 (Dolbeaut–Grotendieck).

$$\overline{\mathcal{H}}^a_b(p) = 0 = \mathcal{H}^b_a(p) \qquad a > 0 \quad p \neq 0.$$

Proof. The result is immediate in the light of general theorems on cohomologies of filtered complexes [18, 24] once it is realized that the filtration (36) is bounded: the filtration degree of the cochains at level N ranges from $-N$ to N. For the sake of completeness it is worth repeating a very simple argument given in [11, 25], which is at the same time a good illustration of how the general technology does work. This argument will also be used in the following.

† In the general scheme [12, 24] of 'perturbative' computation of cohomologies this new gradation defines a filtration of the complex by decreasing the family of subcomplexes.

Any cochain of bidegree $(a, *)$ can be decomposed into a finite sum of homogeneous components with respect to the filtration degree: $\Psi^a_* = \sum_{i\geqslant m} \Psi^a_{*;i}$. The equation $\overline{\mathcal{D}}\Psi^a_* = 0$ written in terms of (37) implies a chain of equations for homogeneous constituents and, in particular, $\overline{\mathcal{D}}_{(0)}\Psi^a_{*;m} = 0$ for the component of lowest filtration degree. Since the cohomologies of $\overline{\mathcal{D}}_{(0)}$ are trivial $\Psi^a_{*;m} = \overline{\mathcal{D}}_{(0)}F^{a-1}_{*;m}$. The lowest filtration component of an equivalent element $\Psi'^a_* = \Psi^a_* - \overline{\mathcal{D}}F^{a-1}_{*;m}$ is of degree at least $m + 1$. The procedure repeated appropriately many times leads to the conclusion that $\Psi^a_* = \overline{\mathcal{D}}\Phi^{a-1}_*$ for some Φ^{a-1}_*. The second equality follows from the Poincaré–Serre duality of lemma 2.1. □

The Dolbeaut–Grotendieck lemma was proved here in a purely algebraic way. It seems to be an interesting question as to whether the result can be obtained by more classical analytic methods [13] with the help of the complex analytic structure of the underlying coset space $\mathit{Diff}(S^1)/S^1$ constructed in [20].

The lemma above implies directly the vanishing theorem for relative cohomology (27) and provides a convenient description of non-vanishing classes at ghost number zero in terms of bigraded cocycles.

Theorem 2.1 (Vanishing theorem).

(1) $\quad H^r_{\mathrm{rel}}(p) = 0 \qquad r \neq 0 \qquad p$ *on-shell*

(2) $\quad H^0_{\mathrm{rel}}(p) \sim \overline{\mathcal{Z}}^0_0(p)/\mathcal{D}\overline{\mathcal{Z}}^0_1(p).$

Proof. (1) Take $\Psi^r \in Z^r_{\mathrm{rel}}(p)$. In the case of $r > 0$, Ψ^r has the following bidegree decomposition: $\Psi^r = \sum_0^m \Phi^{r+i}_i$. Taking into account the Dolbeaut–Grotendieck vanishing lemma for $\overline{\mathcal{H}}^a_b(p)$ and repeating the arguments used in its proof, with $\overline{\mathcal{D}}_{(0)}$ replaced by $\overline{\mathcal{D}}$ and the filtration degree replaced by the anti-ghost number i ,it is not difficult (the property $\mathcal{D}\Psi^a_0 \equiv 0$ has to be used too) to obtain $H^r_{\mathrm{rel}}(p) = 0$ for $r > 0$. The vanishing in the case of $r < 0$ can be obtained in the same way with the use of vanishing of $\mathcal{H}^a_b(p)$; $b > 0$ cohomologies or simply by Poincaré duality for relative cohomologies.

(2) Vanishing of $\overline{\mathcal{H}}^a_b(p)$ and $\mathcal{H}^b_a(p)$ classes with $a > 0$ implies (by the same arguments as above) $Z^0_{\mathrm{rel}}(p) \ni \Psi^0 = \sum_0^m \Phi^i_i \sim \Psi^0_0$ with $\Psi^0_0 \in \overline{\mathcal{Z}}^0_0(p)$ and $\sim$ in the sense of relative cohomology. Further identification of $\overline{\mathcal{Z}}^0_0(p)$ cocycles amounts to $\Psi^0_0 - \Psi'^0_0 = \mathcal{D}F^0_1$ with $\overline{\mathcal{D}}F^0_1 = 0$. □

It is important to note that the statements above are local with respect to the momentum variable as they were proved in the adapted lightcone frame. For the proofs to be pushed through, the non-vanishing of the p^+ momentum component was an inevitable assumption. Note, however, that they can be globalized for massive states as the mass-shell equation excludes $p^\pm$ components taking a value of zero in any lightcone frame. Consequently, the definition of the space of physical states as a direct integral of local cohomology spaces does not cause any problem at massive levels. The question of such a fusion in the cases of topologically non-trivial tachyonic ground shells and the massless cone (first excited level for $d = 25$) is not so straightforward and is left as an open problem.

It is also worth noting that the condition for $\Psi^0_0(p) = \upsilon(p)\otimes\omega$; $\upsilon \in \mathcal{H}(p)$ to be a cocycle reads $0 = \overline{\mathcal{D}}\Psi^0_0(p) = \sum_{m>0} L_m\upsilon(p) \otimes c_{-m}\omega$ and consequently

$$\overline{\mathcal{Z}}^0_0(p) = \mathcal{H}_{\mathrm{phys}}(p) \otimes \omega \tag{38}$$

where $\mathcal{H}_{\mathrm{phys}}(p)$ is the subspace of highest weight vectors (10) of the original covariant space (5) of string states.

Once the relative cohomology at non-zero momentum is known it is possible to reconstruct absolute BRST cohomology spaces (22). It will be shown that the two natural injections $i_\mp : \mathcal{C}^0_{\mathrm{rel}}(p) \to \mathcal{C}^{\mp\frac{1}{2}}(p)$ of relative complex into invariant (on-mass-shell) subspace (25) of full BRST complex: $i_-(\Psi) = \Psi$ and $i_+(\Psi) = c_0\Psi$ induce isomorphisms of the corresponding cohomology spaces. Taken in their raw form they do not transform the relative cocycles into absolute ones: $Dc_0\Psi = -c_0 D_{\mathrm{rel}}\Psi + M\Psi$ (26). There is, however, a convenient representation of relative cohomology classes given in theorem 2.1. In order to obtain proper mappings it is enough to restrict the injections to $\overline{\mathcal{Z}}^0_0$ cocycles all being annihilated by M. Then the induced mappings

$$i^*_\pm : \overline{\mathcal{Z}}^0_0(p)/\mathcal{D}\overline{\mathcal{Z}}^0_1(p) \to H^{\pm 1/2}(p) \qquad i^*_+[\Psi^0_0] = [c_0\Psi^0_0]_{\mathrm{abs}} \qquad i^*_-[\Psi^0_0] = [\Psi^0_0]_{\mathrm{abs}} \tag{39}$$

are well defined on the equivalence classes. For i^*_- it is obvious as $i_-\mathcal{D} = Di_-$ on $\overline{\mathcal{Z}}^0_1$. In the case of i_+ the change of representative $\Psi^0_0 \to \Psi^0_0 + \mathcal{D}z^0_1$ induces $i_+(\Psi^0_0) \to i_+(\Psi^0_0) - D(c_0 z^0_1) + Mz^0_1$. However, Mz^0_1 is a $\overline{\mathcal{D}}$ closed element of bidegree $(1, 0)$. Consequently, it is D_{rel} closed and must be exact: $Mz^0_1 = D_{\mathrm{rel}}F^0 \equiv DF^0$; $F^0 \in \mathcal{C}^0_{\mathrm{rel}}(p)$. Hence i^*_+ is well defined too.

The complete description of absolute cohomologies of the BRST complex must include on-shell states at $p = 0$. They are admissible (23) only for spacetime dimension $d = 25$ and are all located at level $N = 1$. The space $\mathcal{C}_{(0)}(0)$ of on-shell cochains is spanned by

$$\begin{aligned} &\mathcal{C}^{1/2}(0) = c_0 V \oplus \mathbb{C}c_{-1}\omega(0) \qquad && \mathcal{C}^{3/2}(0) = \mathbb{C}c_0 c_{-1}\omega(0) \\ &\mathcal{C}^{-1/2}(0) = V \oplus \mathbb{C}c_0 b_{-1}\omega(0) \qquad && \mathcal{C}^{-3/2}(0) = \mathbb{C}b_{-1}\omega(0) \end{aligned} \tag{40}$$

where $V = \mathbb{C}\{a^\mu_{-1}\omega(0), u_{-1}\omega(0); 0 \leqslant \mu \leqslant 24\}$.

The cohomology spaces (22) of the full BRST complex are described in the following.

Theorem 2.2 (Absolute BRST cohomology).

(1) $$H(p) = H^{-1/2}(p) \oplus H^{1/2}(p) \quad H^{\pm 1/2}(p) = i^*_\pm\left(\overline{\mathcal{Z}}^0_0(p)/\mathcal{D}\overline{\mathcal{Z}}^0_1(p)\right) \quad p \neq 0$$

(2) $$H^{\pm 1/2}(0) \cong i_\pm V \qquad H^{\pm 3/2}(0) \cong \mathcal{C}^{\pm 3/2}(0) \quad d = 25.$$

Proof.

(1) It will be shown first that vanishing of relative cohomology implies $H^r(p) = 0$ for $r \neq \pm\frac{1}{2}$. For $\Psi^{r+\frac{1}{2}} = c_0\Phi^r + \Phi^{r+1}$ the cocycle condition $D\Psi^{r+\frac{1}{2}} = 0$ decays into two equations $D_{\mathrm{rel}}\Phi^r = 0$ and $M\Phi^r + D_{\mathrm{rel}}\Phi^{r+1} = 0$ on its relative components. If $r \neq 0$ one has $\Phi^r = D_{\mathrm{rel}}F^{r-1}$ and $M\Phi^r = MD_{\mathrm{rel}}F^{r-1}$. Consequently, $\Psi^{r+\frac{1}{2}} \sim \Psi^{r+\frac{1}{2}} - D(c_0F^{r-1}) = \Phi^{r+1}$. If $r \neq -1$ then $\Psi^{r+\frac{1}{2}} \sim D_{\mathrm{rel}}F^r = DF^r$ and the statement is evident.

In order to check the isomorphism property of $i^*_\pm$ one should use the closure conditions above for $\Psi^{1/2} = c_0\Phi^0 + \Phi^1$ and $\Psi^{-1/2} = \Phi^0 + c_0\Phi^{-1}$ under the total differential D. They amount to $\Psi^{-1/2} \sim \Phi^0$ and $\Psi^{1/2} = c_0\Phi^0 - D^{-1}_{\mathrm{rel}}M\Phi^0$, where $D_{\mathrm{rel}}\Phi^0 = 0$ and $D^{-1}_{\mathrm{rel}}M\Phi^0$ denoting any solution of the equation $M\Phi^0 = D_{\mathrm{rel}}F^1$. This equation can always be solved as relative cohomologies are trivial at ghost number 2.

Theorem 2.1 guarantees that there exists F^{-1} such that $\Phi^0 = z^0_0 + D_{\mathrm{rel}}F^{-1}$. Hence $\Psi^{1/2} + D(c_0F^{-1}) = c_0 z^0_0$ and $\Psi^{-1/2} \sim \Phi^0 + D(F^{-1}) = z^0_0$ and both maps $i^*_\pm$ are onto.

Injectivity of $i^*_\pm$ is also a consequence of vanishing of relative cohomologies. Assume $i^*_-[\Psi]_{\mathrm{rel}} = i^*_-[\Psi']_{\mathrm{rel}}$, i.e. $\Psi - \Psi' = D\Phi$, $\Phi = c_0F^{-2} + F^{-1}$ which, in particular, gives $D_{\mathrm{rel}}F^{-2} = 0$. Hence $F^{-2} = D_{\mathrm{rel}}F^{-3}$ for some F^{-3} and cocycles are equivalent: $\Psi - \Psi' = D_{\mathrm{rel}}(F^{-1} + MF^{-3})$. Injectivity of i^*_+ can be proved in exactly the same way.

(2) Direct calculation shows that all basis vectors of (40) except of $c_0 b_{-1}\omega(0)$ are closed and, moreover, $c_{-1}\omega(0) = Dc_0 b_{-1}\omega(0)$. From (26) it is clear that closed elements containing c_0 are cohomologically non-trivial and so are their Poincaré duals. □

The result of theorem 2.1 on vanishing of relative cohomology enables one to find simple formulae for dimensions of cohomology spaces $H^{\pm 1/2}(p)$ or isomorphic space $H^0_{\mathrm{rel}}(p)$.

The method relies on the observation [15] that under favourable circumstances these dimensions are equal to the Euler–Poincaré characteristic of the complex.

For the relative complex at fixed on-mass-shell momentum p and level $N = -(1/2\alpha)p^2 + (d-1)/24$

$$\mathcal{C}_{\mathrm{rel}}(p) = \bigoplus_{-r_N}^{r_N} \mathcal{C}^r_{\mathrm{rel}}(p) \qquad r_N = \max\{r;\ r(r+1) \leqslant N\}$$

its Euler–Poincaré characteristic $ch(p)$ satisfies the chain of identities:

$$ch(p) := \sum_{-r_N}^{r_N} (-1)^r \dim \mathcal{C}^r_{\mathrm{rel}}(p) = \sum_{-r_N}^{r_N} (-1)^r \dim H^r_{\mathrm{rel}}(p) = \dim H^0_{\mathrm{rel}}(p). \quad (41)$$

The second equality in the formulae above is nothing but an expression of the Euler–Poincaré principle (which is a quite simple and very general algebraic statement [18, 25] provided the complexes under consideration are of finite dimension), while the third one is a consequence of the triviality of higher cohomologies. In this case the Euler–Poincaré characteristic counts the number of linearly independent physical states on the corresponding mass shells.

These numbers can be encoded in the form of a generating function for the left-hand side of (41):

$$ch_q = q^{(1/2\alpha)p^2 - (d-1)/24} P(q) \qquad \text{such that} \quad ch(p) = ch_q|_0 \quad (42)$$

where $|_0$ denotes the constant term of the Laurent series in q. The power series $P(q)$ is the product of a well known partition function corresponding to $d+1$ families of bosonic excitation operators $P^{d+1}_{\mathrm{str}}(q) = \prod_{n\geqslant 0}(1-q^n)^{-(d+1)}$ and the alternating sum $P_{\mathrm{alt}}(q)$ of the partition functions of fixed ghost number subspaces $\mathcal{C}^r_\infty$. This function can be calculated as the weighted trace of the operator $Tr(q^{\mathcal{L}_0}(-1)^{G_0})$ over the ghost sector $\mathcal{C}_\infty$ as in [15, 16]. It can also be obtained from the generating series in two variables

$$P_{\mathrm{gh}}(q,t) = \prod_{n>0}(1+tq^n)\prod_{n>0}(1+t^{-1}q^n) = \sum_{N\geqslant 0} q^N \left(\sum_{-r_N}^{r_N} P^r_N t^r \right) \quad (43)$$

where the powers of q keep track of the level, while the coefficients of the powers of t count the number of states at fixed level and of corresponding ghost number. Hence the alternating sums are generated by $P_{\mathrm{alt}}(q) = P_{\mathrm{gh}}(q,-1) = \prod_{n>0}(1-q^n)^2$ and finally

$$\dim H^0_{\mathrm{rel}}(p) = q^{(1/2\alpha)p^2 - (d-1)/24} \prod_{n\geqslant 0}(1-q^n)^{-d+1}\Big|_0. \quad (44)$$

The generating function above indicates that the states at levels $N = -(1/2\alpha)p^2 + (d-1)/24$ are generated by $d-1$ families of bosonic-type excitation operators.

The formulae in (44) will be useful for the proof of the no-ghost theorem in the next section.

3. No-ghost theorem and cohomology representations

The no-ghost theorem can be proved by at least two methods. A direct method can be applied in the case where the subspace of cohomology representatives (gauge slice) is known explicitly. Then the proof of the no-ghost theorem consists in checking that the scalar product induced on this subspace is positive. A virtue of the second method [15] is that the only data it uses are the vanishing of relative cohomology and the Lorenzian character of the scalar product in the covariant space of string states (5). It also has an important advantage for the logic of the reasoning here. It will be demonstrated that the positivity of the scalar product on cohomologies allows for a straightforward identification of cohomologically trivial states in (5) and for constructions of gauge slices.

The scalar product in the space of a relative complex induces a pairing in the space of relative cohomology classes via their representatives:

$$\langle [\Psi(p)]_{\rm rel}, [\Psi'(p)]_{\rm rel}\rangle := \big(\Psi(p), c_0\Psi'(p)\big). \tag{45}$$

It is well defined, i.e. does not depend on the choice of representatives due to the fact that $D_{\rm rel}$ is self-adjoint with respect to $(\,,)$. The Poincaré duality guarantees that the pairing is non-degenerate. Since the physical states of the string are already identified with the relative cohomology classes (theorem 2.1) the no-ghost theorem is equivalent to the statement that $\langle\,,\rangle$ of (45) is positive.

The proof of positivity relies on the observation that the no-ghost theorem is equivalent to the statement that the Euler–Poincaré characteristic of the complex is equal to its signature, provided the Euler–Poincaré principle applies to the last one.

The signature of the complex endowed with some Hermitian scalar product $(\,,)$ is defined as the difference between the number of positive eigenvalues and the number of negative eigenvalues in its diagonal form. In the case of a relative string complex it is possible† to prove:

$$\operatorname{sign} H^0_{\rm rel}(p) = \operatorname{sign}\mathcal{C}^0_{\rm rel}(p) = \operatorname{sign}\mathcal{C}_{\rm rel}(p). \tag{46}$$

It is then clear that the scalar product (45) on cohomology spaces is positively defined if and only if

Theorem 3.1 (No-ghost theorem).

$$\dim H^0_{\rm rel} = \operatorname{sign} H^0_{\rm rel}.$$

Proof. In order to prove the equality above it is most convenient to compare the generating function (44) with that for signatures:

$$\operatorname{sign}_q := q^{(1/2\alpha)p^2-(d-1)/24}S(q) \qquad \text{such that} \qquad \operatorname{sign}\mathcal{C}^0(p) = \operatorname{sign}_q\big|_0.$$

The key observation for the construction of the generating series $S(q)$ is that the signatures are multiplicative with respect to the tensor product. Hence $S(q)$ is a product of $S_{\rm str}(q)$ describing the signatures of string excitation space and $S_{\rm gh}(q)$ corresponding to that of the ghost sector.

The signatures of the string sector are determined by the Lorenzian character of string modes (6) with a timelike oscillator contributing -1 when appearing in an odd power. The contributions of this timelike sector to the signatures are described by the function which counts

† A simple proof based on the Hodge–Serre construction of the positive scalar product and the harmonic representation of cohomology classes may be found in [16] or [26].

the difference between even and odd partitions of a level number and is given in terms of a bosonic partition function of two variables,

$$P_{\mathrm{str}}(q,t) := \prod_{n>0}(1-tq^n)^{-1} = \sum_{N\geqslant 0} q^N \left(\sum_{k>0} P_N^k t^k\right)$$

with $t=-1$. The final result is obtained by multiplying the above by the partition function of the remaining d oscillators:

$$S_{\mathrm{str}}(q) = \prod_{n>0}(1-q^n)^{-d}(1+q^n)^{-1}.$$

In order to compute $S_{\mathrm{gh}}(q)$ it is enough to note (the second equality in (46)) that only the excitations of $c_{-n}b_{-n}$ doublets of weight $2n$ contribute to the signature: +1 when the number of doublets is even and -1 when it is odd. From the generating function for even fermionic partitions analogous to (43) of the previous section:

$$S_{\mathrm{gh}}(q) = \prod_{n>0}(1+tq^{2n})|_{t=-1} = \prod_{n>0}(1-q^{2n}).$$

Hence the dimension and the signature generating functions are equal. □

The no-ghost theorem allows for a straightforward identification of the subspace $\mathcal{D}\overline{\mathcal{Z}}^0_1(p)$ of exact cocycles in $\overline{\mathcal{Z}}^0_0(p)$. The space $\overline{\mathcal{Z}}^0_0(p)$ was already identified (38) with the set of highest weight vectors of (5). The characterization of exact elements is given in the following:

Lemma 3.1.

$$\Psi^0_0(p) \in \mathcal{D}\overline{\mathcal{Z}}^0_1(p) \qquad \textit{if and only if} \quad (\Psi^0_0(p), \cdot\,)_{\mathrm{rel}} = 0.$$

Proof. The right pointed implication follows from $(\mathcal{D}\,\cdot\,,\,\cdot)_{\mathrm{rel}} = -(\,\cdot\,,\overline{\mathcal{D}}\,\cdot\,)_{\mathrm{rel}}$.

Assume $(\Psi^0_0(p),\,\cdot\,)_{\mathrm{rel}} = 0$ and $[\Psi^0_0(p)]_{\mathrm{rel}} \neq 0$. Then $\langle[\Psi^0_0(p)]_{\mathrm{rel}}, H^0_{\mathrm{rel}}\rangle = 0$ which contradicts the no-ghost theorem. □

It follows from the lemma above that the set of exact cocycles coincides with the subspace of null highest weight vectors of (10). This observation allows for simple constructions of the representatives of cohomology classes.

The total space (10) as well as the total space $\overline{\mathcal{Z}}^0_0$ is defined according to the decomposition (24) of the on-mass-shell complex:

$$\overline{\mathcal{Z}}^0_0 = \bigoplus_{N\geqslant 0}\int_{S_N} \mathrm{d}\mu^N(p)\,\overline{\mathcal{Z}}^0_0(p). \tag{47}$$

The discussion of the kinematics of the states from (47) splits naturally into two qualitatively different cases: the tachyonic vacuum states ($N=0$) and generically massive excited states. Only in the case of $d=25$ is the first excited level massless.

All vacuum states are parametrized by the momenta belonging to the mass shells defined by $p^2 = 2\alpha(d-1)/24$ which are of non-trivial topological type S^{d-2}.

The massive shells S_N are most conveniently parametrized in terms of lightcone coordinates associated with some fixed lightcone frame $\{k_\pm, e_i;\ \ i = 1\ldots d-2\}$. The disjoint components $S_N^\pm$ of S_N are distinguished by the sign of never vanishing lightcone coordinate $p^+ = kp \in \mathbb{R}\setminus\{0\}$. The momenta are then described by p^+ and the set of $d-2$ transverse coordinates $p^i = e_i p$. The complementary lightcone component $p^- = k_- p$

becomes dependent and equals $p^- = (1/2p^+)(\overline{p}^2 + m_N^2)$ with mass squared given in (23) and $\overline{p}$ denoting the transverse part of the momentum.

In the lightcone coordinates the measure restricted to the connected component S_*^+ of S_* present in (47) acquires a particularly simple and level-independent form:

$$\mathrm{d}\mu(p) = \frac{\mathrm{d}p^+}{p^+}\,\mathrm{d}^{d-2}\overline{p}. \tag{48}$$

As in the standard lightcone constructions one may choose the lightcone coordinate $x^+ := k_+x$ as the evolution parameter and the mass-shell condition is then solved by fixing the lightcone time dependence of the states from (47):

$$\Psi^{(N)}(\overline{p}, p^+, x^+) = \mathrm{e}^{\mathrm{i}x^+(1/2p^+)(\overline{p}^2+m_N^2)}\Psi^{(N)}(\overline{p}, p^+). \tag{49}$$

The pairing of the states (49) has to be evaluated at an arbitrarily chosen but fixed moment of time. It is also assumed that the scalar product of the vectors located at different momenta is proportional to $p^+\delta(p^+ - p'^+)\delta(\overline{p} - \overline{p}')$.

The description of the space $\overline{\mathcal{Z}}_0^0(p) \simeq \mathcal{H}_{\mathrm{phys}}(p)$ can be given in terms of a so-called spectrum-generating algebra of DDF operators [27] associated with a fixed lightcone frame $\{k_\pm, e_i;\ i = 1\ldots d-2\}$. Their construction is briefly reported in the appendix. The DDF algebra is generated by the set of elementary excitation operators consisting of $d-2$ families of transverse modes (A1) $\{A_m^i\}_{m\in\mathbb{Z}}; i = 1,\ldots,d-2$, the family of Liouville modes (A3) $\{U_m\}_{m\in\mathbb{Z}}$ and a set of Brower [4] operators (A2) $\{B_m\}_{m\in\mathbb{Z}}$ to describe the excitations along the k_- lightcone direction. They are well defined on all states with non-vanishing lightcone component $p^+ := k_+p$ of the momentum and their commutation relation reads

$$[A_m^i, A_n^j] = m\delta_{n+n}\delta^{ij} \qquad [U_m, U_n] = m\delta_{n+m} \qquad [B_m, B_n] = (m-n)B_{m+n}. \tag{50}$$

The above operators are designed [4] to commute with all quantum constraints (7). Their additional property of primary importance is that all have a definite weight with respect to the covariant level operator R_{str} of (9): $[R_{\mathrm{str}}, \mathcal{O}_m] = -m\mathcal{O}_m$. Since they commute with L_0 they must shift the energy of the state consistently: $p \to p + m\alpha\frac{1}{p^+}k^+$. It was proved in [1] that the state $\Psi(p)$ with on-mass-shell momentum p and at corresponding level N belongs to $\mathcal{H}_{\mathrm{phys}}(p)$ if and only if

$$\Psi(p) = \vartheta^N(A, B, U)\omega\left(p + \alpha\frac{1}{p^+}Nk_+\right) \tag{51}$$

where $\vartheta^N(\cdot)$ denotes a polynomial in DDF creation operators of level N. From the property of longitudinal zero mode $B_0\omega(p) = 0$ and the commutation relations (50) it follows that the state (51) is null, i.e. cohomologically trivial, if and only if the corresponding polynomial contains any longitudinal excitation operator $B_{-m};\ m > 0$. Hence the subspace of $\overline{\mathcal{Z}}_0^0(p)$:

$$\mathcal{H}_{\mathrm{LC}}(p) := \{\Psi(p);\ B_0\Psi(p) = 0\} \tag{52}$$

is transverse to $\mathcal{D}\overline{\mathcal{Z}}_1^0(p)$ and defines a good section of the space $\overline{\mathcal{Z}}_0^0(p)$ over the quotient $\overline{\mathcal{Z}}_0^0(p)/\mathcal{D}\overline{\mathcal{Z}}_1^0(p)$. The space (52) does not contain longitudinally excited states and it is natural to call it the lightcone gauge slice. In the lightcone parametrization (49) the condition (52) acquires the form of the Schrödinger equation:

$$B_0\Psi(p^+, \overline{p}, x^+) \equiv \left(\mathrm{i}\frac{1}{\alpha}p^+\frac{\partial}{\partial x^+} - L_0(A, U)\right)\Psi(p^+, \overline{p}, x^+) = 0 \tag{53}$$

where $L_0(A, U)$ is the standard expression in transverse and Liouville DDF modes given in (A5). According to (53) and (52) the time-dependent momentum eigenstates from $\mathcal{H}_{\mathrm{LC}}(p)$ are generated by polynomials in transverse DDF operators acting on an appropriate tachyonic vacuum:

$$\mathcal{H}^N_{\mathrm{LC}}(p^+, \overline{p}) \ni \Psi(p^+, \overline{p}, x^+) = \vartheta^N(A(x^+), U(x^+))\omega(p^+, \overline{p}, x^+). \tag{54}$$

The operators generating the states from $\mathcal{H}_{\mathrm{LC}}(p)$ and vacuum vectors exhibit a very simple lightcone time (x^+) dependence:

$$\begin{aligned} &A^i_m(x^+) = \mathrm{e}^{\mathrm{i}m\alpha(x^+/p^+)}a^i_m \qquad U_m(x^+) = \mathrm{e}^{\mathrm{i}m\alpha(x^+/p^+)}u_m \\ &\omega(p^+, \overline{p}, x^+) = \mathrm{e}^{(\mathrm{i}x^+/2p^+)(\overline{p}^2+m_0^2)}\omega(p^+, \overline{p}) \\ &a^i_m = A^i_m(0) \qquad u_m = U_m(0) \qquad \omega(p^+, \overline{p}) = \omega(p^+, \overline{p}, 0). \end{aligned} \tag{55}$$

The time evolution of the excitation operators above is an automorphism of their CCR algebra. It is this crucial property that allows one to describe the time dependence in the factorized form of (54) and, furthermore, to describe the model in terms of initial data for the quantum mechanical Schrödinger equation (53) in the Fock space generated by the algebra of a^i_m and u_m. It is then natural to introduce the space $\mathcal{H}_{\mathrm{lc}}$ which contains all vectors from $\mathcal{H}_{\mathrm{LC}}$ evaluated at $x^+ = 0$. The space $\mathcal{H}_{\mathrm{lc}}$ can be thought of as an abstract Fock space independent of the explicit realization of its generators in terms of DDF operators. Hence,

$$H^0_{\mathrm{rel}} \simeq \mathcal{H}_{\mathrm{lc}} \simeq \mathcal{H}^{\mathrm{vac}} \oplus \bigoplus_{N>0} \int \frac{\mathrm{d}p^+}{p^+}\, \mathrm{d}^{d-2}\overline{p}\, \mathcal{H}^N_{\mathrm{lc}}(p)$$

where $\mathcal{H}^{\mathrm{vac}}$ denotes the Hilbert space of tachyonic ground states which cannot be consistently described within the framework of the lightcone gauge.

There is another interesting section of cohomology classes in the spacetime dimensions $d < 25$. In all of these cases (in contrast to $d = 25$ as the corresponding Virasoro module is not free) the Fock-type Liouville excitations of the states from (52) are equally well described in terms of the vectors of Verma modules generated by

$$L_n(U) = \tfrac{1}{2}\sum_{m\in\mathbb{Z}} : U_{-m}U_{m+n} : + 2\mathrm{i}\sqrt{\beta}\, U_n + 2\beta\delta_{n0}.$$

Consequently, the transformation

$$\mathcal{H}_{\mathrm{LC}}(p) \ni \vartheta^N(A, L(U))\omega\left(p + \alpha\frac{1}{p^+}Nk_+\right) \mapsto \vartheta^N(A, B^{\mathrm{NG}})\omega\left(p + \alpha\frac{1}{p^+}Nk_+\right) \in \mathcal{H}_{\mathrm{NG}}(p)$$

where $B^{\mathrm{NG}}_m = L_m(U) + B_m$, is a shift along null directions and defines an equivalent section of cohomology classes. It is natural to call this section the Nambu–Goto gauge slice: the states in $\mathcal{H}_{\mathrm{NG}}(p)$ do not contain Liouville excitations and are generated by the algebra of transverse DDF modes and B^{NG}_m operators satisfying

$$[B^{\mathrm{NG}}_m, B^{\mathrm{NG}}_n] = (m-n)B^{\mathrm{NG}}_{n+m} + \tfrac{1}{12}m(m^2-1)(26-d)\delta_{m+n}$$

which is exactly the longitudinal Virasoro algebra of the non-critical Nambu–Goto string [4]. The description of $\mathcal{H}_{\mathrm{NG}}(p)$ within the framework of lightcone parametrization of mass shells is more difficult than in the case of (52) since all B^{NG}_m are first-order differential operators with respect to the lightcone time variable x^+.

Concluding remarks

There are at least two interesting problems which are left open.

It was shown in [1] that in addition to the string model, defined by the critical values (21) of Liouville coupling and the intercept parameter, there are families of unitary models with longitudinal degrees of freedom. They fall into a continuous series:

$$a \leqslant 1 \qquad 0 < \beta \leqslant \frac{24 - D}{48} \tag{56}$$

or a discrete one:

$$\begin{aligned} \beta &= \frac{24 - D}{48} + \frac{1}{8m(m+1)} && m \geqslant 2 \\ a &= 1 - \frac{((m+1)r - ms)^2 - 1}{4m(m+1)} && 1 \leqslant r \leqslant m - 1 \qquad 1 \leqslant s \leqslant r. \end{aligned} \tag{57}$$

It is an interesting question as to whether the string models of non-critical series do admit the resolutions in terms of some complex of BRST type. It is tempting to conjecture that an appropriate complex may be defined as the kernel of the curvature operator (20) D^2. It can be immediately noted that this subcomplex consists of cochains with non-negative ghost number.

The second problem is the global description of the lightcone gauge slice. The Fock space picture of the space of physical states is admissible over the open subsets of the ground tachyonic shell defined as the injective images of massive shells under the shift mappings introduced in (51). For obvious reasons this local description in terms of lightcone Fock space is not relativistic invariant. The generators of the Lorentz group constructed in [2] on the lightcone gauge slice are self-adjoint only on the subspaces of massive states with vacuum vectors excluded. It is then important to understand the global geometry of the space of string states which are only locally visible as Fock excitations.

Acknowledgments

The author would like to thank Professor Anatol Odzijewicz from the Institute of Theoretical Physics of the University in Białystok for his helpful hand and a great heart. Support from Professors Jan Łopuszanski and Jerzy Lukierski from the University of Wrocław was at least as important. The author would also like to thank his younger colleague Marcin Daszkiewicz for his interest and discussions. This work was supported in part by KBN grant 2P03 A 012 19.

Appendix

In order to define the DDF operators one introduces the conformal fields

$$X^\mu(z) = \sqrt{\alpha}x^\mu - \mathrm{i}\frac{1}{\sqrt{\alpha}}P^\mu \log(z) + \sum_{m\neq 0}\frac{\mathrm{i}}{m}a_m^\mu z^{-m} \qquad \varphi(z) = \sum_{m\neq 0}\frac{\mathrm{i}}{m}u_m z^{-m}$$

$$P^\mu(z) = \mathrm{i}z\left(\frac{\mathrm{d}}{\mathrm{d}z}X^\mu\right)(z) = \sum_{m\in\mathbb{Z}} a_m^\mu z^{-m} \qquad \pi(z) = \mathrm{i}z\left(\frac{\mathrm{d}}{\mathrm{d}z}\varphi\right)(z) = \sum_{m\in\mathbb{Z}} u_m z^{-m}$$

with standard commutation rules [28].

The construction of DDF operators is performed in some fixed lightcone frame in the momentum space $\{k_\pm, e_i\}_{i=1}^{d-2}$. The transverse DDF modes

$$A_m^i(k_+) = \frac{1}{2\pi \mathrm{i}} \oint \frac{\mathrm{d}z}{z} : e_i P(z) \exp \frac{\mathrm{i}m\sqrt{\alpha}X^+(z)}{p^+} : \qquad i = 1, \ldots, d-2 \tag{A1}$$

and the longitudinal ones

$$\widetilde{B}_m^-(k_+) = \frac{1}{2\pi \mathrm{i}} \oint \frac{\mathrm{d}z}{z} : \left(\frac{p^+}{\sqrt{\alpha}} P^-(z) - \frac{m}{2} z \frac{\mathrm{d}}{\mathrm{d}z} \log \left(\frac{\sqrt{\alpha} P^+(z)}{p^+} \right) \right) \exp \frac{\mathrm{i}m\sqrt{\alpha}X^+(z)}{p^+} : . \tag{A2}$$

are given by slightly modified expressions with respect to those of [4, 27]. The virtue of this rather simple modification is that the domain of the operators under consideration is extended to the open subset $p^+ \neq 0$ of the momentum space, while in the original constructions it was restricted to some integral lattice.

The family of DDF operators corresponding to the original Liouville u modes is defined by

$$U_m(k_+) = \frac{1}{2\pi \mathrm{i}} \oint \frac{\mathrm{d}z}{z} : \left(\pi(z) - 2\mathrm{i}\sqrt{\beta} z \frac{\mathrm{d}}{\mathrm{d}z} \log \left(\frac{\sqrt{\alpha} P^+(z)}{p^+} \right) \right) \exp \frac{\mathrm{i}m\sqrt{\alpha}X^+(z)}{p^+} : \tag{A3}$$

where $\beta = (25 - d)/48$. The operators (A1) and (A3) satisfy canonical commutation rules of elementary modes, while the commutation relations of longitudinal modes (A2) are that of Virasoro algebra with the central charge $c = 24$. They neither commute with transverse DDF operators nor with Liouville ones.

For this reason it is convenient to introduce the shifted Brower modes:

$$B_n = \widetilde{B}_n^- - L_n(A, U) + \delta_{n0} \tag{A4}$$

where

$$L_n(A, U) = \tfrac{1}{2} \sum_{m \in \mathbb{Z}} : A_{-m} A_{n+m} : + \tfrac{1}{2} \sum_{m \in \mathbb{Z}} : U_{-m} U_{n+m} : + 2\mathrm{i}\sqrt{\beta} U_n + 2\beta \delta_{n0}. \tag{A5}$$

The operators (A4) commute with (A1) and (A3) and their structural relations reads

$$[B_m, B_n] = (m - n) B_{m+n}.$$

References

[1] Hasiewicz Z and Jaskólski Z 1996 *Nucl. Phys.* B **464** 85
[2] Daszkiewicz M, Hasiewicz Z and Jaskólski Z 1998 *Nucl. Phys.* B **514** 437
[3] Goto T 1971 *Prog. Theor. Phys.* **46** 1560
Godard P, Goldstone J, Rebbi C and Thorn C B 1973 *Nucl. Phys.* B **56** 109
[4] Brower R C 1972 *Phys. Rev.* D **6** 1655
[5] Rebbi C 1974 *Phys. Rep.* **12** 1
[6] Mandelstam S 1974 *Nucl. Phys.* B **83** 413
[7] Witten E 1986 *Nucl. Phys.* B **268** 253
Friedan D, Martinec E and Shenker S 1986 *Nucl. Phys.* B **271** 93
Hata H, Itoh K, Kugo T, Kunimoto H and Ogawa K 1986 *Phys. Rev.* D **34** 2360
[8] Hwang S 1983 *Phys. Rev.* D **28** 2614
[9] Marnelius R 1983 *Nucl. Phys.* B **211** 14
[10] Bilal A 1992 *Phys. Lett.* B **282** 309
[11] Bouwknegt P, McCarthy J and Pilch K 1992 *Commun. Math. Phys.* **145** 54
[12] Griffiths P and Harris J 1978 *Principles of Algebraic Geometry* (New York: Wiley)
[13] Wells R O Jr 1973 *Differential Analysis on Complex Manifolds* (Englewood Cliffs, NJ: Prentice Hall)
Chern S-S 1979 *Complex Manifolds Without Potential Theory* (New York: Springer)

[14] Kato M and Ogawa K 1983 *Nucl. Phys.* B **212** 443
Kugo T and Ojima I 1979 *Suppl. Prog. Theor. Phys.* **66** 1
Itoh K 1990 *Nucl. Phys.* B **342** 449
[15] Spigelglass M 1987 *Nucl. Phys.* B **283** 205
[16] Figueroa-O'Farril J M and Kimura T 1989 *Commun. Math. Phys.* **124** 105
[17] Frenkel I B, Garland H, Zukerman G J 1986 *Proc. Nat. Acad. Sci., USA* **83** 8442
[18] Greub W, Halperin S and Vanstone R 1976 *Connctions, Curvature and Cohomology* vol 3 (New York: Academic)
[19] Banks T and Peskin M E 1986 *Nucl. Phys.* B **264** 513
[20] Kirillov A A and Yuriev D V 1988 *J. Geom. Phys.* **5** 351
[21] Kirillov A A and Yuriev D V 1986 *Funct. Anal. Appl.* **20** 79 (in Russian)
Kirillov A A and Yuriev D V 1987 *Funct. Anal. Appl.* **21** 35 (in Russian)
[22] Hilton P J and Stammbach U 1970 *A Course in Homological Algebra* (Berlin: Springer)
[23] Lian B and Zuckermann G J 1989 *Commun. Math. Phys.* **125** 301
[24] Bott R and Tu L W 1982 *Differential Forms in Algebraic Topology* (Berlin: Springer)
[25] Dubrovin B A, Novikov S P and Fomenko A T 1984 *Methods of Homology Theory* (Moscow: Nauka) (in Russian)
[26] Miscenko A S 1984 *Vector Bundles and Applications* (Moscow: Nauka) (in Russian)
[27] Del Giudice E, Di Vecchia P and Fubini S 1972 *Ann. Phys., NY* **70** 378
[28] Goddard P and Olive D I 1986 *Int. J. Mod. Phys.* A **1** 303